Complex Numbers Review for EE-201 (Hadi saadat)

In our numbering system, *positive* numbers correspond to distance measured along a horizontal line to the right. *Negative* numbers are represented to the left of origin. Numbers corresponding to distances along the line shown in Figure 1 are called real numbers and the horizontal axis is known as the real axis.

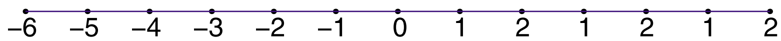

Figure 1. The system of real numbers.

Consider a point c in a two-dimensional plane located a distance M along a line at an angle θ, taken counterclockwise from the positive horizontal axis.

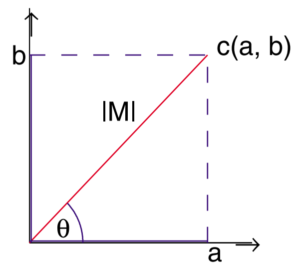

Figure 2. Graphical representation of a point in the x-y plane (known as complex plane).

The projection of c along the horizontal axis or the so called real axis is shown by a. This component is known as the real component of c. The projection on the vertical axis is shown by b. Thus, we may show c by its two coordinates as $c(a, b)$. To differentiate between the two components, it is unfortunately common practice to call the vertical component of c the imaginary component. In this context, the vertical axis is known as the imaginary axis.

Operator j

Consider a circle of radius unity with center at the origin of the x-y plane as shown in Figure 3.

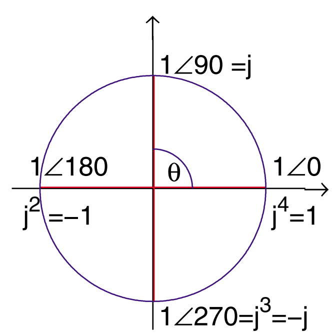

Figure 3. Graphical representation of the operator j.

The radius along the positive real axis may be written as $1\angle 0$. If this line of unity length is rotated counterclockwise by 90°, it will be located along the vertical axis, i.e., $1\angle 90^\circ$. This operation is represented by the operator j, and we write

$$j = 1\angle 90^\circ$$

This means that if any real number is multiplied by j it is rotated by $90°$ in a counterclockwise direction.

Rotating through another 90, we have

$$(j)(j) = (1\angle 90°)(1\angle 90°) = 1\angle 180° = -1$$

i.e.,

$$j^2 = -1$$

Taking the square root, we may write

$$j = \sqrt{-1}$$

The equation $j^2 = -1$ is a statement that the two *operations* are equivalent. However, and that is the beauty of the concept, in all algebraic computation imaginary numbers can be handled as if j had a numerical value.

Continuing the rotation by another $90°$, i.e., a total of $270°$, we write

$$j^3 = (j^2)(j) = -j = 1\angle 270° = 1\angle -90°$$

and

$$j^4 = (j^2)(j^2) = (-1)(-1) = 1\angle 360° = 1\angle 0°$$

We can write the reciprocal operation $\frac{1}{j}$ as

$$\frac{1}{j} = \frac{(j)}{(j)(j)} = \frac{j}{j^2} = \frac{j}{-1} = -j = 1\angle -90°$$

Rectangular and Polar Forms

With the introduction of the operator j, the point $c(a, b)$ in Figure 2 may be represented as

$$c = a + jb$$

This representation indicates that the real part, a, is measured along the real axis (the abscissa) and the so called imaginary part, b, is reckoned along the imaginary axis (the ordinate). This representation is known as the *rectangular form* of a complex number c.

The complex number $c = a + jb$ may also be represented as the length or magnitude of a line segment, $|M|$, at an angle, θ, as indicated in Figure 2. Thus,

$$c = a + jb = |M|\angle\theta$$

The form $c = |M|\angle\theta$ is called the *polar form*, $|M|$ is the magnitude and θ is called the *angle* or the *argument* of c. The conversion from rectangular to polar form can be deduced immediately from Figure 2 in conjunction with the Pythagorean theorem, i.e.,

$$|M| = \sqrt{a^2 + b^2}$$

and the angle is given by

$$\theta = \tan^{-1}\frac{b}{a}$$

For conversion from polar to rectangular from,

$$a = |M|\cos\theta$$

$$b = |M|\sin\theta$$

Thus, c can be written as

$$c = |M|(\cos\theta + j\sin\theta)$$

The Euler's identity – Exponential Form

Consider a complex number with the Magnitude $|M|$ = 1,

$$c = 1\angle\theta = \cos\theta + j\sin\theta$$

Taking derivative of c with respect to θ, result in

$$\frac{dc}{d\theta} = -\sin\theta + j\cos\theta = j(\cos\theta + j\sin\theta)$$

or

$$\frac{dc}{d\theta} = jc$$

Separating the variable,

$$\frac{1}{c}dc = jd\theta$$

Integrating the above equation, we get

$$\ln c = j\theta + K$$

where K is the constant of integration. The magnitude of c is unity regardless of the angle, therefore, at $\theta = 0$, $\ln(1) = j(0) + K$, or $K = 0$, and

$$\ln c = j\theta$$

or

$$c = e^{j\theta}$$

With c given by the equation $c = \cos\theta + j\sin\theta$, the *exponential form* also known as the *Euler's identity* is

$$e^{j\theta} = \cos\theta + j\sin\theta$$

for an angle $-\theta$, we obtain

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Adding and subtracting the above two equations lead to the representation for $\cos\theta$ and $\sin\theta$,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

With the addition of the Euler's identity, the three way of representing a complex number, rectangular, polar and exponential forms ar all equivalent and we may write

$$c = a + jb = |M|\angle\theta = |M|e^{j\theta}$$

Mathematical Operations

The conjugate of a complex number $c = a + jb$ denoted by c^* and is defined as

$$c^* = a - jb$$

or in polar form for $c = |M|\angle\theta$, we have

$$c^* = |M|\angle{-\theta}$$

To add or subtract two complex numbers, we add (or subtract) their real parts and their imaginary parts. For two complex numbers designated by $c_1 = a_1 + jb_1$ and $c_2 = a_2 + jb_2$, their sum is

$$c_1 + c_2 = (a_1 + a_2) + j(b_1 + b_2)$$

The multiplication of c_1 and c_2 in rectangular form is obtained as follows (note $j^2 = -1$)

$$c_1c_2 = (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 - b_1b_2 + j(a_1b_2 + b_1a_2)$$

or in polar form for $c_1 = |M_1|\angle\theta_1$, and $c_2 = |M_2|\angle\theta_2$, The multiplication of c_1 and c_2 is

$$\begin{aligned} c_1c_2 &= (|M_1|\angle\theta_1)(|M_2|\angle\theta_2) \\ &= (|M_1|e^{j\theta_1})(|M_2|e^{j\theta_2}) \\ &= |M_1||M_2|e^{j(\theta_1+\theta_2)} \\ &= |M_1||M_2|\angle\theta_1 + \theta_2 \end{aligned}$$

if a complex number $c = a + jb$ is multiplied by its conjugate $c^* = a - jb$, the result is

$$cc^* = (a + jb)(a - jb) = a^2 + b^2 = |M|^2$$

or in polar form

$$cc^* = |M|\angle\theta|M|\angle{-\theta} = |M|^2$$

For the division of c_1 by c_2 in rectangular form, we multiply numerator and denominator by the conjugate of the denominator, this results in

$$\begin{aligned} \frac{c_1}{c_2} &= \frac{(a_1 + jb_1)}{(a_2 + jb_2)} \\ &= \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{a_1a_2 + b_1b_2}{{a_2}^2 + {b_2}^2} + j\frac{b_1a_2 - a_1b_2}{{a_2}^2 + {b_2}^2} \end{aligned}$$

or in polar form for $c_1 = |M_1|\angle\theta_1$, and $c_2 = |M_2|\angle\theta_2$, the division of c_1 by c_2 is

$$\begin{aligned} \frac{c_1}{c_2} &= \frac{|M_1|\angle\theta_1}{|M_2|\angle\theta_2} \\ &= \frac{|M_1|e^{j\theta_1}}{|M_2|e^{j\theta_2}} \\ &= \frac{|M_1|}{|M_2|}e^{j(\theta_1-\theta_2)} \\ &= \frac{|M_1|}{|M_2|}\angle\theta_1 - \theta_2 \end{aligned}$$

Example 1

For the complex numbers
$c_1 = 20\angle 36.87°$, $c_2 = 40\angle{-53.13°}$, $c_3 = 40 + j80$, $c_4 = 12\angle\frac{\pi}{6}$, $c_5 = 5\angle{-\frac{\pi}{6}}$, and $c_6 = 30 + j40$
Find $c = (c_1 + c_2 + c_3)/(c_4 * c_5 - c_6)$
Substituting for the values and converting c_1 and c_2 to rectangular form, we have

$$\begin{aligned} c &= \frac{20\angle 36.87° + 40\angle{-53.13°} + 40 + j80}{(12\angle\frac{\pi}{6})(5\angle{-\frac{\pi}{6}}) - (30 + j40)} \\ &= \frac{(16 + j12) + (24 - j32) + (40 + j80)}{(60 + j0) - (30 + j40)} = \frac{80 + j60}{30 - j40} \\ &= \frac{100\angle 36.87°}{50\angle{-53.13°}} = 2\angle 90° = j2 \end{aligned}$$

Use MATLAB To evaluate the complex number c described in Example 1. In MATLAB, we use the following commands:

```
c1 = 20*exp(j*36.87*pi/180);  % In MATLAB angles must be in radian
c2 = 40*exp(-j*53.13*pi/180);
c3 = 40 + j*80;
c4 = 12*exp(j*pi/6);
c5 = 5*exp(-j*pi/6);
c6 = 30 + j*40;
c = (c1+c2+c3)/(c4*c5-c6)
M=abs(c)                    % Magnitude
theta = angle(c)*180/pi     % Angle in degree
```

Save in a file with extension m, and run to get the result

```
c =
        0.0000 + 2.0000i
M =
        2.0000
theta =
        90.0000
```

Example 2

A complex function is described by

$$g = \frac{2500(j\omega)}{(25 + j\omega)(100 + j\omega)}$$

Write an script m-file to evaluate the magnitude and phase angle of g for ω from 0 to 200 in step of 1, and obtain the magnitude and phase angle plots versus ω.

We use the following statements:

```
w=0:1:200;
g= (2500*j*w)./((25+j*w).*(100+j*w)); %Array Multiplication & division use .* & ./
M=abs(g);                             % Magnitude
theta = angle(g)*180/pi;              % Angle in degree
subplot(2,1,1), plot(w, M), grid
ylabel('M'), xlabel('\omega')
subplot(2,1,2), plot(w, theta), grid
ylabel('\theta, degree'), xlabel('\omega')
```

The result is

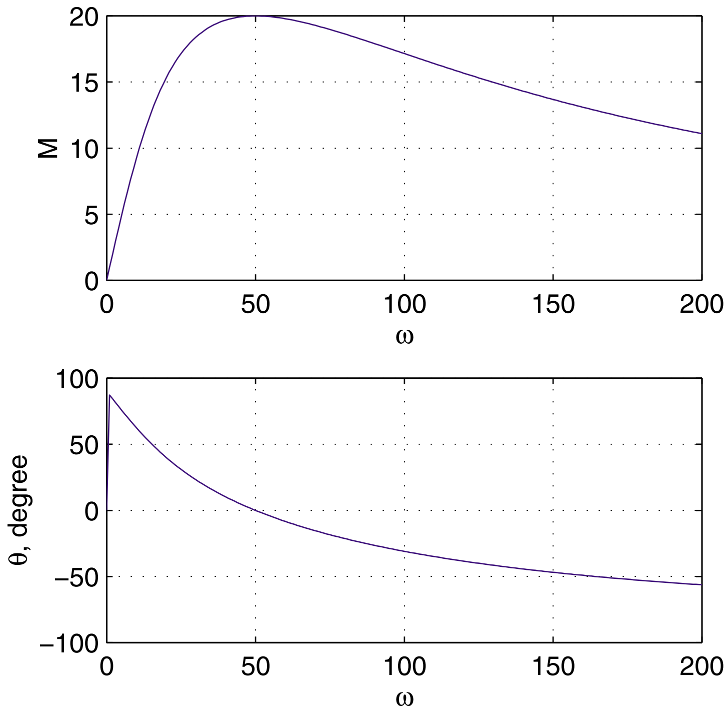

Figure 4. Magnitude and phase angle plots for complex function in Example 2.

Homework Problems

1. Using your calculator evaluate g for the function in Example 2 for (a) $\omega = 11$, (b) $\omega = 50$, and (c) $\omega = 112$. Express your answers both in rectangular and polar forms.

2. A complex function is described by

$$g = \frac{10000}{(10 + j\omega)(100 - \omega^2 + 10(j\omega))}$$

Write an script m-file to evaluate the magnitude and phase angle of g for ω from 0 to 50 in step of 1, and obtain the magnitude and phase angle plots versus ω.

3. Using your calculator evaluate g for the function in Problem 2 for (a) $\omega = 10$, (b) $\omega = 14.142$, and (c) $\omega = 21.5$. Express your answers both in rectangular and polar forms.

Prepared for EE-201 by H. Saadat, October 17, 99