Review (Formulas)

Normalized Signal Energy and Signal Power

CT periodic signals with period T

$$E = \int_{-\infty}^{\infty} |x^2(t)|dt \qquad P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x^2(t)dt$$

DT periodic signals with period N

$$E = \sum_{-N}^{N} |x^2[n]| \qquad P = \frac{1}{2N}\sum_{-N}^{N} |x^2[n]|$$

Fourier series

The trigonometric form of the Fourier series for a periodic signal $x(t)$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

where

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\, dt \qquad n \neq 0$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t\, dt \qquad n \neq 0$$

The exponential form of the Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

where

$$X_n = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}\, dt$$

X_n in terms of a_n, and b_n are

$$X_n = \begin{cases} \frac{1}{2}(a_n - jb_n),\ n > 0 \\ \frac{1}{2}(a_n + jb_n),\ n < 0 \end{cases}$$

$$X_0 = a_0$$

Parseval's theorem gives the average power in terms of the line spectra

$$P_{av} = \sum_{n=-\infty}^{\infty} |X_n|^2$$

$$= X_0^2 + 2\sum_{n=1}^{\infty} |X_n|^2$$

Simplification for Fourier series coefficient due to signal symmetry

Signal Symmetry	**Signal property**	X_n **coefficients**	a_n & b_n **coefficients**
Even	$x(t) = x(-t)$	Real	$b_n = 0$, for all n
Odd	$x(t) = -x(-t)$	Imaginary	$a_n = 0$, for all n
Half-wave even	$x\left(t \pm \frac{T_0}{2}\right) = x(t)$	Complex $X_n = 0, n$ odd	$a_n = b_n = 0$, n odd
Half-wave odd	$x\left(t \pm \frac{T_0}{2}\right) = -x(t)$	Complex $X_n = 0, n$ even	$a_n = b_n = 0$, n even

Convolution

The response of a LTI system to an input $x(t)$ is expressed as a *Convolution integral* is

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \quad \text{or}$$
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

where $h(t)$ is the impulse response.

Impulse response

The impulse response of

$$\frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) \quad \text{with} \quad y(t) = 0$$

is equivalent to solving the homogeneous equation

$$\frac{dy(t)}{dt} + a_0 y(t) = 0 \quad \text{with} \quad y(0) = b_0$$

Similarly, the impulse response of a second order system

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

can be found as the solution to the homogeneous equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = 0 \quad \text{with}$$

$$y(0) = 0 \;\text{ and}\, \frac{dy(0)}{dt} = b_0$$

From s-domain $h(t) = \mathcal{L}^{-1}H(s)$

Fourier Transform Fourier transform and the inverse Fourier transform are

$$\mathcal{F}\{x(t)\} = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$\mathcal{F}^{-1}\{X(f)\} = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Fourier Transform Pairs

$x(t)$	$X(f)$
$\dot{\delta}(t)$	$j2\pi f$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j2\pi ft_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$
1	$\delta(f)$
$\Pi\left(\frac{t}{\tau}\right)$	$\tau\, sinc(\tau f)$
$2W\, sinc(2Wt)$	$\Pi\left(\frac{f}{2W}\right)$
$\Lambda\left(\frac{t}{\tau}\right)$	$\tau\, sinc^2(\tau f)$
$sinc(t)$	$\Pi(f)$
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$sgn(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\, sgn(f)$
$e^{-\alpha t}u(t)$	$\frac{1}{\alpha + j2\pi f}$
$te^{-\alpha t}u(t)$	$\frac{1}{(\alpha + j2\pi f)^2}$
$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
$e^{-\pi\left(\frac{t}{\tau}\right)^2}$	$e^{-\pi\left(\frac{f}{\tau}\right)^2}$
$\cos 2\pi f_0 t$	$\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$
$\sin 2\pi f_0 t$	$\frac{1}{2j}\delta(f-f_0) - \frac{1}{2j}\delta(f+f_0)$
$tu(t)$	$\frac{j}{4\pi}\dot{\delta}(f) - \frac{1}{4\pi^2 f^2}$
$\hat{x}(t) = x(t) * \frac{1}{\pi t}$	$-j\, sgn(f)\, X(f)$
$\sum_{m=-\infty}^{\infty} \delta(t - mT_s)$	$\frac{1}{T_s}\sum_{m=-\infty}^{\infty} \delta(f - mf_s)$

Energy Spectral Density

The normalized energy of a signal in terms of its Fourier signal is

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

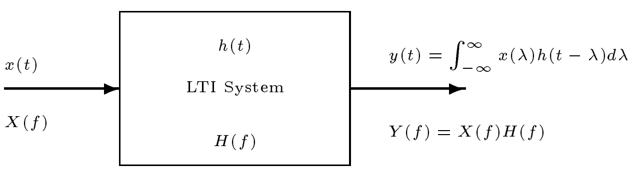
The Energy Spectral Density is

$$G(f) = |X(f)|^2$$

Fourier Transform Theorems (Operational Transforms)

Operational Properties of the Fourier Transform		
Transform	$x(t)$	$X(f)$
Duality	$X(t)$	$x(-f)$
Superposition	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(f) + a_2X_2(f)$
Time delay	$x(t - t_0)$	$X(f)e^{-j2\pi ft_0}$
Time reversal	$x(-t)$	$X(-f)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
Frequency translation	$x(t)e^{j2\pi f_0t}$	$X(f - f_0)$
Modulation	$x(t)\cos(2\pi f_0t)$	$\frac{1}{2}X(f - f_0) + \frac{1}{2}X(f + f_0)$
Derivative	$\frac{dx(t)}{dt}$	$j2\pi fX(f)$
	$\frac{d^nx(t)}{dt^n}$	$(j2\pi f)^nX(f)$
	$\frac{1}{-j2\pi}\frac{d}{dt}X(f)$	$tx(t)$
Integral	$\int_{-\infty}^{\infty} x(t)dt$	$\frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$
Convolution	$x_1(t) * x_2(t)$	$X_1(f)X_2(f)$
Conjugation	$x^*(t)$	$X^*(f)$
Multiplication	$x_1(t) * x_2(t)$	$X_1(f) * X_2(f)$

System Analysis with Fourier transform
The response of a LTI system is represented as



The time domain response is the inverse Fourier transform of $Y(f)$, i.e.,

$$y(t) = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

The system transfer function H(f) can be found in following ways:

1. From the impulse response $h(t)$, find $H(f) = \mathcal{F}[h(t)]$.

2. Obtain Fourier transform of the system differential equation and find the ratio $\frac{Y(f)}{X(f)}$ to find $H(f)$.

3. Transform the time domain equations into the phasor domain, expressing circuit elements with
impedances and find the transfer function.

4. Find the Laplace transform of the differential equations, obtain the s-domain transfer function and replace s by $j2\pi f$.

Steady-State Response to Harmonic Inputs
The Fourier transform of the output of LTI system is

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n||H(nf_0)|e^{j(2\pi nf_0 t + \angle X_n + \angle H(nf_0))}$$

or

$$y(t) = a_0 H(0) + \sum_{n=1}^{\infty} a_n |H(nf_0)| \cos(n\omega_0 t + \angle H(nf_0)) + \sum_{n=1}^{\infty} b_n |H(nf_0)| \sin(n\omega_0 t + \angle H(nf_0))$$

Butterworth Lowpass Filters

First-order $\quad H(s) = \dfrac{\omega_c}{s + \omega_c}$

Second-order $\quad H(s) = \dfrac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$

Third-order $\quad H(s) = \dfrac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}$

The Frequency response amplitude of the nth order LP Butterworth Filter is

$$H(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

Sampling

The sampled signal is given by $x_s(t) = p(t)x(t)$

The spectrum $X_s(f)$ of the sampled signal is

$$X_s(f) = \sum_{-\infty}^{\infty} C_n X(f - nf_s)$$

where C_n is the Fourier series coefficients of $p(t)$. For impulse-train sampling, $C_n = f_s$, and the spectrum of the sampled signal becomes

$$X_s(f) = f_s \sum_{-\infty}^{\infty} X(f - nf_s)$$

Signal Recovery – Ideal interpolation

The spectrum of the recovered signal by an ideal low pass filter (bandwidth $0.5f_s$, and height $\frac{1}{f_s}$) is given by $X(f) = \frac{1}{f_s}X_1$, and the recovered signal is

$$y(t) \approx x(t) = \sum_{k=-\infty}^{\infty} x(kT)h(t - KT)$$

where $h(t)$ is the LPF impulse response, for ideal LPF $h(t)$ is the $Sinc$ function.

Quantization and Encoding

The quantizing step size S, for q quantizing level ($q = 2^n$) is given by

$$S = \frac{D}{q} = \frac{D}{2^n}$$

where D is the maximum variation of $x(t)$.

The mean square error (Signal noise) is $= \frac{D^2}{12}2^{-2n}$.

If P_s is the signal power, the signal-to-power ratio ($SNR = P_s/E$) in decibels is

$$\text{SNR}_{dB} = 10.79 + 6.02n + 10\log_{10} P_s - 20\log_{10} D$$

Discrete-Time Fourier Series (DTFS)

The trigonometric Fourier series coefficients for a DT signal is follows:

$$a_0 = \frac{2}{N}\sum_{k=0}^{N-1} x_k$$

$$a_n = \frac{2}{N}\sum_{k=0}^{N-1} x_k \cos\left(\frac{2\pi n}{N}k\right) \qquad n \neq 0$$

$$b_n = \frac{2}{N}\sum_{k=0}^{N-1} x_k \sin\left(\frac{2\pi n}{N}k\right) \qquad n \neq 0$$

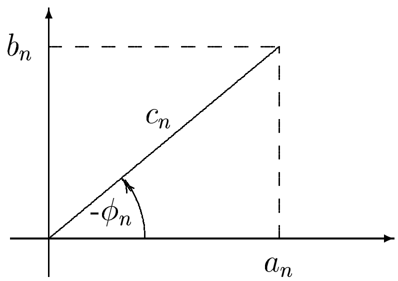
The Fourier series expression for $x(t)$ is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\frac{N}{2}} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

or

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\frac{N}{2}} c_n \cos(n\omega_0 t + \phi_n)$$

where c_n and ϕ_n can be found from the following triangle



Trigonometric Identities

$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\sin u \cos v = \frac{1}{2}[\sin(u-v) + \sin(u+v)]$$