

Recurrence Relations

Consider the following recursive function:

```
int nfact(unsigned int n)
{
    if (n == 0 || n == 1)
        return 1;
    else
        return n * nfact(n-1);
}
```

The above function can have its run time expressed as the following recurrence relation:

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 1 \\ O(1) + T(n-1) & \text{if } n > 1 \end{cases}$$

$$\begin{aligned} T(n) &= O(1) + T(n-1) && \text{if } n > 1 \\ \text{But, } T(n-1) &= O(1) + T(n-2) && \text{if } n > 2 \\ \text{Thus, } T(n) &= O(1) + O(1) + T(n-2) && \text{if } n > 2 \\ \text{So, } T(n) &= 2 \cdot O(1) + T(n-2) && \text{if } n > 2 \end{aligned}$$

$$\begin{aligned} \text{Now, } T(n-2) &= O(1) + T(n-3) && \text{if } n > 3 \\ \text{Thus, } T(n) &= 3 \cdot O(1) + T(n-3) && \text{if } n > 3 \end{aligned}$$

$$\begin{aligned} T(n-3) &= O(1) + T(n-4) && \text{if } n > 4 \\ \text{So, } T(n) &= 4 \cdot O(1) + T(n-4) && \text{if } n > 4 \end{aligned}$$

$$\begin{aligned} T(n-4) &= O(1) + T(n-5) && \text{if } n > 5 \\ \text{So, } T(n) &= 5 \cdot O(1) + T(n-5) && \text{if } n > 5 \end{aligned}$$

In general, the recurrence can be expressed as:

$$T(i) = i \cdot O(1) + T(n-i) \quad \text{if } n > i$$

Realize, i = number of recursive calls. Therefore, $i = n-1$.

Substituting $n-1$ for i into the equation above yields:

$$T(n-1) = (n-1) \cdot O(1) + T(n-(n-1)) \quad \text{if } n > n-1$$

Simplifying

$$T(n) = n \cdot O(1) + T(1)$$

For the first time, we have expressed $T(n)$ non-recursively (no $T(n)$ term on the right hand side of the equation). So we can now solve the recurrence.

Since $T(1)$ is $O(1)$ and $n \cdot O(1)$ is $O(n)$ We have $T(n) = O(n)$