

Supplementary notes for EE-201 (H. Saadat)

POWER IN SINGLE-PHASE AC CIRCUITS

Figure 1 shows a single-phase sinusoidal voltage supplying a load.

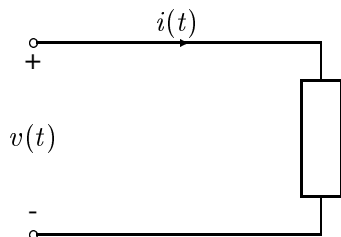


FIGURE 1
Sinusoidal source supplying a load.

Let the instantaneous voltage be

$$v(t) = V_m \cos(\omega t + \theta_v) \quad (1)$$

and the instantaneous current be given by

$$i(t) = I_m \cos(\omega t + \theta_i) \quad (2)$$

The instantaneous power $p(t)$ delivered to the load is the product of voltage $v(t)$ and current $i(t)$ given by

$$p(t) = v(t) i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i) \quad (3)$$

It is informative to write (3) in another form using the trigonometric identities, the result is

$$p(t) = \underbrace{|V||I| \cos \theta [1 + \cos 2(\omega t + \theta_v)]}_{p_R(t)} + \underbrace{|V||I| \sin \theta \sin 2(\omega t + \theta_v)}_{p_X(t)} \quad (4)$$

Energy flow into the circuit Energy borrowed and returned by the circuit

where θ is the angle between voltage and current, or the impedance angle. θ is positive if the load is inductive, (i.e., current is lagging the voltage) and θ is negative if the load is capacitive (i.e., current is leading the voltage). $|V| = V_m/\sqrt{2}$ and $|I| = I_m/\sqrt{2}$.

For $V_m = 1$, $I_m = 0.8$, $\theta_v = 0$, and $\theta_i = -\pi/3$ (inductive load), we use the following commands to plot $v(t)$, $i(t)$, and $p(t)$, $p_r(t)$, and $p_x(t)$.

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wt=0:.05:3*pi;
Vm = 1; Im = 0.8;
thetav=0; thetai = -pi/3; theta=thetav-thetai;
v = Vm*cos(wt); i = Im*cos(wt + thetai);
p = v.*i;
pr = Vm*Im/2*cos(theta)*(1+cos(2*(wt+thetav)));
px = Vm*Im/2*sin(theta)*sin(2*(wt+thetav));
subplot(2, 1, 1),plot(wt, v, wt, i, wt, p), grid
title('v(t) = V_m*cos\omegat, i(t) = I_m*cos(\omegat - 60)')
xlabel('\omega t, rad/s')
text(1.4, .85, 'i'), text(3.5, .75, 'p'), text(.1, .85, 'v')
subplot(2, 1, 2),plot(wt, pr, wt, px), grid
text(.8, .45, 'p_x'), text(2.3, .45, 'p_r')
xlabel('\omega t, rad/s')

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The result is shown in Figure 2.

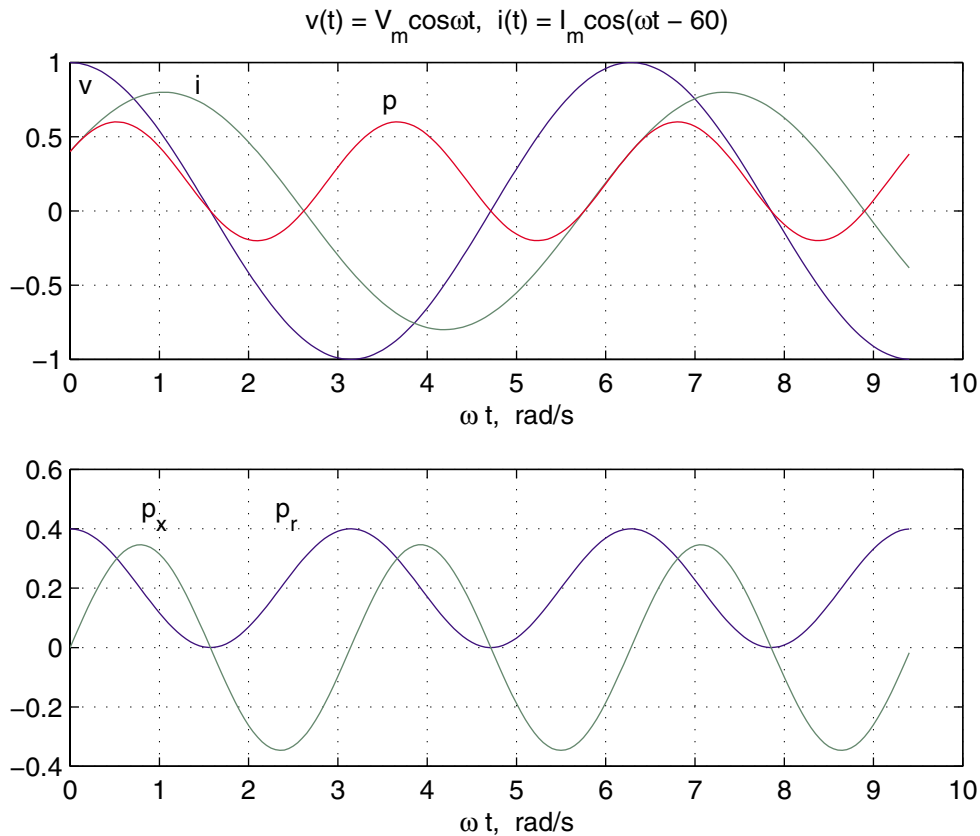


Figure 2 plots of $v(t)$, $i(t)$, $p(t)$, $p_r(t)$, and $p_x(t)$ for inductive load $\theta = \pi/3$.

$$p_R(t) = |V||I| \cos \theta + |V||I| \cos \theta \cos 2(\omega t + \theta_v) \quad (5)$$

The second term in (5), which has a frequency twice that of the source, accounts for the sinusoidal variation in the absorption of power by the resistive portion of the load. Since the average value of this sinusoidal function is zero, the average power delivered to the load is given by

$$P = |V||I| \cos \theta \quad (6)$$

This is the power absorbed by the resistive component of the load and is also referred to as the *active power* or *real power*. The product of the rms voltage value and the rms current value $|V||I|$ is called the *apparent power* and is measured in units of volt ampere. The product of the apparent power and the cosine of the angle between voltage and current yields the real power. Because $\cos \theta$ plays a key role in the determination of the average power, it is called *power factor*. When the current lags the voltage, the power factor is considered lagging. When the current leads the voltage, the power factor is considered leading.

The second component of (4)

$$p_X(t) = |V||I| \sin \theta \sin 2(\omega t + \theta_v) \quad (7)$$

pulsates with twice the frequency and has an average value of zero. This component accounts for power oscillating into and out of the load because of its reactive element (inductive or capacitive). The amplitude of this pulsating power is called *reactive power* and is designated by Q .

$$Q = |V||I| \sin \theta \quad (8)$$

Both P and Q have the same dimension. However, in order to distinguish between the real and the reactive power, the term “var” is used for the reactive power (var is an acronym for the phrase “volt-ampere reactive”). For an inductive load, current is lagging the voltage, $\theta = (\theta_v - \theta_i) > 0$ and Q is positive; whereas, for a capacitive load, current is leading the voltage, $\theta = (\theta_v - \theta_i) < 0$ and Q is negative.

The plot of instantaneous current $i(t)$, voltage, $v(t)$, powers $p(t)$, $p_r(t)$, and $p_x(t)$, for a capacitive load with $\theta_i = \pi/3$, $\theta = \theta_v - \theta_i = -\pi/3$ is shown in Figure 3.

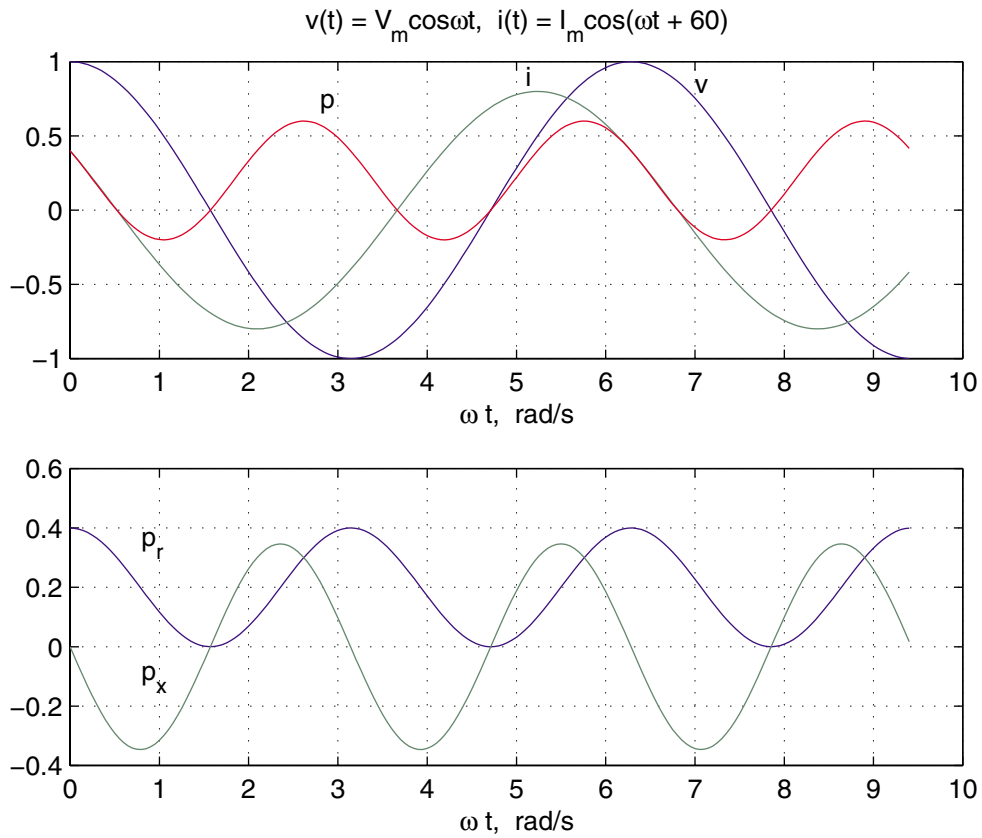


Figure 3 plots of $v(t)$, $i(t)$, $p(t)$, $p_r(t)$, and $p_x(t)$ for capacitive load $\theta = -\pi/3$.

The plots for a purely resistive load, i.e., $\theta = 0$ is shown in Figure 4.

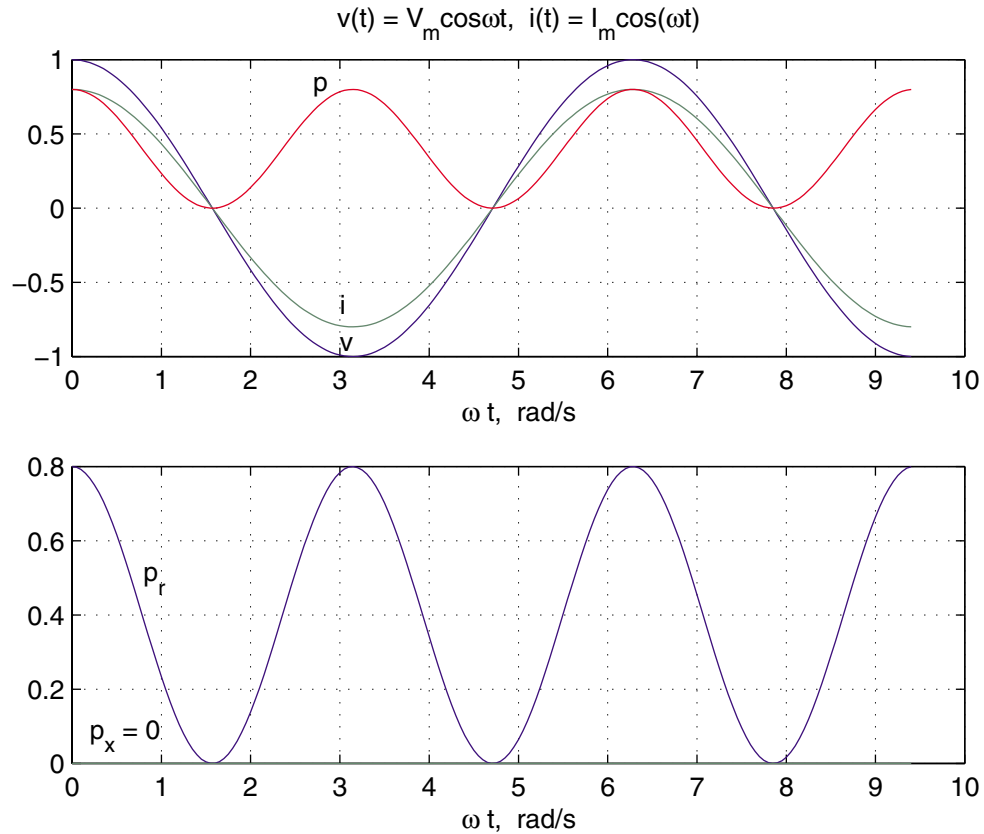


Figure 4 plots of $v(t)$, $i(t)$, $p(t)$, $p_r(t)$, and $p_x(t)$ for a purely resistive load $\theta = 0$.

A careful study of Equations (5) and (7) reveals the following characteristics of the instantaneous power.

- For a pure resistor, the impedance angle is zero and the power factor is unity (UPF), so that the apparent and real power are equal. The electric energy is transformed into thermal energy.
- If the circuit is purely inductive, the current lags the voltage by 90° and the average power is zero. Therefore, in a purely inductive circuit, there is no transformation of energy from electrical to nonelectrical form. The instantaneous power at the terminal of a purely inductive circuit oscillates between the circuit and the source. When $p(t)$ is positive, energy is being stored in the magnetic field associated with the inductive elements, and when $p(t)$ is negative, energy is being extracted from the magnetic fields of the inductive elements.

- If the load is purely capacitive, the current leads the voltage by 90° , and the average power is zero, so there is no transformation of energy from electrical to nonelectrical form. In a purely capacitive circuit, the power oscillates between the source and the electric field associated with the capacitive elements.

COMPLEX POWER

The rms voltage phasor of (1) and the rms current phasor of (2) shown in Figure 5 are

$$V = |V|\angle\theta_v \text{ and } I = |I|\angle\theta_i$$

The term VI^* results in

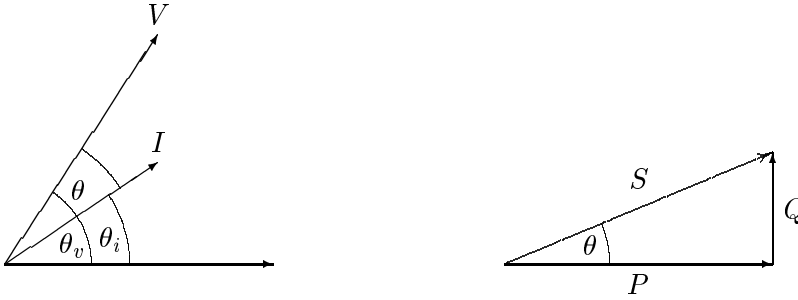


FIGURE 5
Phasor diagram and power triangle for an inductive load (lagging PF).

$$\begin{aligned} VI^* &= |V||I|\angle\theta_v - \theta_i = |V||I|\angle\theta \\ &= |V||I|\cos\theta + j|V||I|\sin\theta \end{aligned}$$

The above equation defines a complex quantity where its real part is the average (real) power P and its imaginary part is the reactive power Q . Thus, the complex power designated by S is given by

$$S = VI^* = P + jQ \quad (9)$$

The magnitude of S , $|S| = \sqrt{P^2 + Q^2}$, is the apparent power; its unit is volt-amperes and the larger units are kVA or MVA. Apparent power gives a direct indication of heating and is used as a rating unit of power equipment. Apparent power has practical significance for an electric utility company since a utility company must supply both average and apparent power to consumers.

The reactive power Q is positive when the phase angle θ between voltage and current (impedance angle) is positive (i.e., when the load impedance is inductive, and I lags V). Q is negative when θ is negative (i.e., when the load impedance is capacitive and I leads V) as shown in Figure 6.

In working with Equation (9) it is convenient to think of P , Q , and S as forming the sides of a right triangle as shown in Figures 5 and 6.

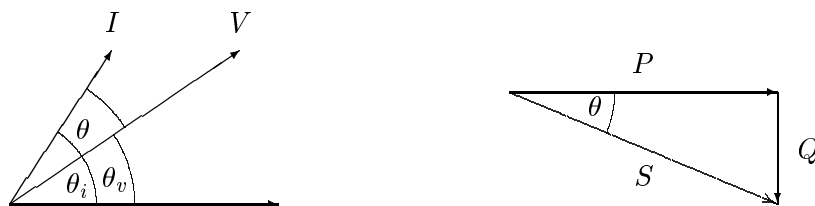


FIGURE 6
Phasor diagram and power triangle for a capacitive load (leading PF).

If the load impedance is Z then

$$V = ZI \quad (10)$$

substituting for V into (9) yields

$$S = VI^* = ZII^* = R|I|^2 + jX|I|^2 \quad (11)$$

From (11) it is evident that complex power S and impedance Z have the same angle. Because the power triangle and the impedance triangle are similar triangles, the impedance angle is sometimes called the *power angle*.

Similarly, substituting for I from (10) into (9) yields

$$S = VI^* = \frac{VV^*}{Z^*} = \frac{|V|^2}{Z^*} \quad (12)$$

From (12), the impedance of the complex power S is given by

$$Z = \frac{|V|^2}{S^*} \quad (13)$$

THE COMPLEX POWER BALANCE

From the conservation of energy, it is clear that real power supplied by the source is equal to the sum of real powers absorbed by the load. At the same time, a balance between the reactive power must be maintained. Thus the total complex power

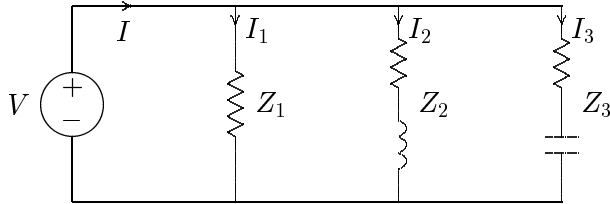


FIGURE 7
Three loads in parallel.

delivered to the loads in parallel is the sum of the complex powers delivered to each. Proof of this is as follows:

For the three loads shown in Figure 7, the total complex power is given by

$$S = VI^* = V[I_1 + I_2 + I_3]^* = VI_1^* + VI_2^* + VI_3^* \quad (14)$$

Example 2.2

In the above circuit $V = 1200\angle 0^\circ$ V, $Z_1 = 60 + j0 \Omega$, $Z_2 = 6 + j12 \Omega$ and $Z_3 = 30 - j30 \Omega$. Find the power absorbed by each load and the total complex power.

$$I_1 = \frac{1200\angle 0^\circ}{60\angle 0} = 20 + j0 \text{ A}$$

$$I_2 = \frac{1200\angle 0^\circ}{6 + j12} = 40 - j80 \text{ A}$$

$$I_3 = \frac{1200\angle 0^\circ}{30 - j30} = 20 + j20 \text{ A}$$

$$S_1 = VI_1^* = 1200\angle 0^\circ(20 - j0) = 24,000 \text{ W} + j0 \text{ var}$$

$$S_2 = VI_2^* = 1200\angle 0^\circ(40 + j80) = 48,000 \text{ W} + j96,000 \text{ var}$$

$$S_3 = VI_3^* = 1200\angle 0^\circ(20 - j20) = 24,000 \text{ W} - j24,000 \text{ var}$$

The total load complex power adds up to

$$S = S_1 + S_2 + S_3 = 96,000 \text{ W} + j72,000 \text{ var}$$

Alternatively, the sum of complex power delivered to the load can be obtained by first finding the total current.

$$\begin{aligned} I &= I_1 + I_2 + I_3 = (20 + j0) + (40 - j80) + (20 + j20) \\ &= 80 - j60 = 100\angle -36.87^\circ \text{ A} \end{aligned}$$

and

$$\begin{aligned} S &= VI^* = (1200\angle 0^\circ)(100\angle 36.87^\circ) = 120,000\angle 36.87^\circ \text{ VA} \\ &= 96,000 \text{ W} + j72,000 \text{ var} \end{aligned}$$

A final insight is contained in Figure 8, which shows the current phasor diagram and the complex power vector representation.

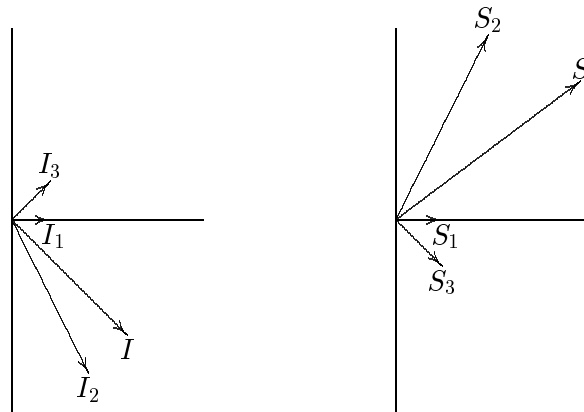


FIGURE 8
Current phasor diagram and power plane diagram.

The complex powers may also be obtained directly from (13)

$$\begin{aligned} S_1 &= \frac{|V|^2}{Z_1^*} = \frac{(1200)^2}{60} = 24,000 \text{ W} + j0 \\ S_2 &= \frac{|V|^2}{Z_2^*} = \frac{(1200)^2}{6 - j12} = 48,000 \text{ W} + j96,000 \text{ var} \\ S_3 &= \frac{|V|^2}{Z_3^*} = \frac{(1200)^2}{30 + j30} = 24,000 \text{ W} - j24,000 \text{ var} \end{aligned}$$

POWER FACTOR CORRECTION

It can be seen from (6) that the apparent power will be larger than P if the power factor is less than 1. Thus the current I that must be supplied will be larger for $PF < 1$ than it would be for $PF = 1$, even though the average power P supplied is the same in either case. A larger current cannot be supplied without additional cost to the utility company. Thus, it is in the power company's (and its customer's) best interest that major loads on the system have power factors as close to 1 as

possible. In order to maintain the power factor close to unity, power companies install banks of capacitors throughout the network as needed. They also impose an additional charge to industrial consumers who operate at low power factors. Since industrial loads are inductive and have low lagging power factors, it is beneficial to install capacitors to improve the power factor. This consideration is not important for residential and small commercial customers because their power factors are close to unity.

Example 2.3

Two loads $Z_1 = 100 + j0 \Omega$ and $Z_2 = 10 + j20 \Omega$ are connected across a 200-V rms, 60-Hz source as shown in Figure 9.

(a) Find the total real and reactive power, the power factor at the source, and the total current.

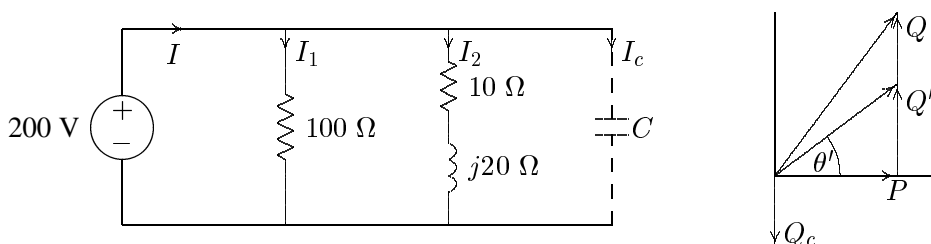


FIGURE 9
Circuit for Example 2.3 and the power triangle.

$$I_1 = \frac{200\angle 0^\circ}{100} = 2\angle 0^\circ \text{ A}$$

$$I_2 = \frac{200\angle 0^\circ}{10 + j20} = 4 - j8 \text{ A}$$

$$S_1 = VI_1^* = 200\angle 0^\circ(2 - j0) = 400 \text{ W} + j0 \text{ var}$$

$$S_2 = VI_2^* = 200\angle 0^\circ(4 + j8) = 800 \text{ W} + j1600 \text{ var}$$

Total apparent power and current are

$$S = P + jQ = 1200 + j1600 = 2000\angle 53.13^\circ \text{ VA}$$

$$I = \frac{S^*}{V^*} = \frac{2000\angle -53.13^\circ}{200\angle 0^\circ} = 10\angle -53.13^\circ \text{ A}$$

Power factor at the source is

$$PF = \cos(53.13) = 0.6 \text{ lagging}$$

(b) Find the capacitance of the capacitor connected across the loads to improve the overall power factor to 0.8 lagging.

Total real power $P = 1200$ W at the new power factor 0.8 lagging. Therefore

$$\begin{aligned}\theta' &= \cos^{-1}(0.8) = 36.87^\circ \\ Q' &= P \tan \theta' = 1200 \tan(36.87^\circ) = 900 \text{ var} \\ Q_c &= 1600 - 900 = 700 \text{ var} \\ Z_c &= \frac{|V|^2}{S_c^*} = \frac{(200)^2}{j700} = -j57.14 \ \Omega \\ C &= \frac{10^6}{2\pi(60)(57.14)} = 46.42 \ \mu\text{F}\end{aligned}$$

The total power and the new current are

$$\begin{aligned}S' &= 1200 + j900 = 1500\angle 36.87^\circ \\ I' &= \frac{S'^*}{V^*} = \frac{1500\angle -36.87^\circ}{200\angle 0^\circ} = 7.5\angle -36.87^\circ\end{aligned}$$

Note the reduction in the supply current from 10 A to 7.5 A.

Example 2.4

Three loads are connected in parallel across a 1400-V rms, 60-Hz single-phase supply as shown in Figure 10.

Load 1: Inductive load, 125 kVA at 0.28 power factor.

Load 2: Capacitive load, 10 kW and 40 kvar.

Load 3: Resistive load of 15 kW.

(a) Find the total kW, kvar, kVA, and the supply power factor.

An inductive load has a lagging power factor, the capacitive load has a leading power factor, and the resistive load has a unity power factor.

For Load 1:

$$\theta_1 = \cos^{-1}(0.28) = 73.74^\circ \text{ lagging}$$

The load complex powers are

$$S_1 = 125\angle 73.74 \text{ kVA} = 35 \text{ kW} + j120 \text{ kvar}$$

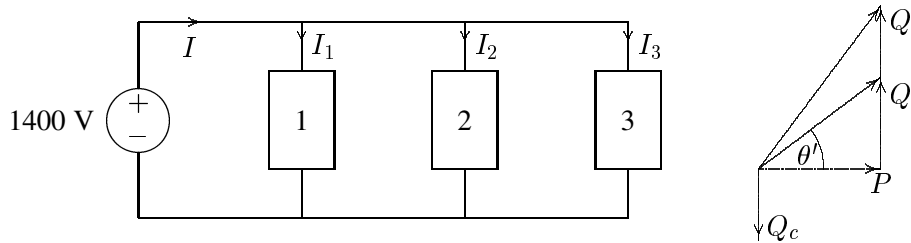


FIGURE 10
Circuit for Example 2.4.

$$S_2 = 10 \text{ kW} - j40 \text{ kvar}$$

$$S_3 = 15 \text{ kW} + j0 \text{ kvar}$$

The total apparent power is

$$\begin{aligned} S &= P + jQ = S_1 + S_2 + S_3 \\ &= (35 + j120) + (10 - j40) + (15 + j0) \\ &= 60 \text{ kW} + j80 \text{ kvar} = 100 \angle 53.13 \text{ kVA} \end{aligned}$$

The total current is

$$I = \frac{S^*}{V^*} = \frac{100,000 \angle -53.13^\circ}{1400 \angle 0^\circ} = 71.43 \angle -53.13^\circ \text{ A}$$

The supply power factor is

$$PF = \cos(53.13) = 0.6 \text{ lagging}$$

(b) A capacitor of negligible resistance is connected in parallel with the above loads to improve the power factor to 0.8 lagging. Determine the kvar rating of this capacitor and the capacitance in μF .

Total real power $P = 60 \text{ kW}$ at the new power factor of 0.8 lagging results in the new reactive power Q' .

$$\theta' = \cos^{-1}(0.8) = 36.87^\circ$$

$$Q' = 60 \tan(36.87^\circ) = 45 \text{ kvar}$$

Therefore, the required capacitor kvar is

$$Q_c = 80 - 45 = 35 \text{ kvar}$$

and

$$X_c = \frac{|V|^2}{S_c^*} = \frac{1400^2}{j35,000} = -j56 \Omega$$
$$C = \frac{10^6}{2\pi(60)(56)} = 47.37 \mu\text{F}$$

and the new current is

$$I' = \frac{S'^*}{V^*} = \frac{60,000 - j45,000}{1400\angle 0^\circ} = 53.57\angle -36.87^\circ \text{ A}$$

Note the reduction in the supply current from 71.43 A to 53.57 A.